

Elementary deformations and the hyperKähler-quaternionic Kähler correspondence

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Abstract The hyperKähler-quaternionic Kähler correspondence constructs quaternionic Kähler metrics from hyperKähler metrics with a rotating circle symmetry. We discuss how this may be interpreted as a combination of the twist construction with the concept of elementary deformation, surveying results of our forthcoming paper. We outline how this leads to a uniqueness statement for the above correspondence and indicate how basic examples of c-map constructions may be realised in this context.

1 Introduction

The twist construction was introduced in [16, 17] as a geometric construction that reproduces T-duality arguments in the physicists literature for geometries with torsion. It has proved successful in constructing compact simply connected examples of a number of classes of non-Kähler geometries. However, elsewhere in the physics literature string theory dualities are used to construct metrics of special holonomy. In particular, the c-map construction of Cecotti et al. [6] produces quaternionic Kähler metrics from projective special Kähler manifolds. An intermediate stage in this construction is a passage from hyperKähler manifolds of a given dimension to quaternionic Kähler manifolds of the same dimension.

HyperKähler and quaternionic Kähler metrics are two of the infinite families of geometries in the holonomy classification of Berger [4, 5]. They are both Einstein geometries and there are many open questions about their structure and classifica-

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tion. In 2008, Haydys [9] showed how to each quaternionic Kähler manifold with circle action one may associate hyperKähler manifolds with a symmetry that fixes only one of the complex structures. He also provided a description of how to invert that construction. Later Hitchin [11] gave a twistor interpretation of this construction along the lines of [12], and [3, 2] provided expressions in arbitrary signature. The metric constructions here all have the flavour of making a conformal change, but with two different factors along and orthogonal to directions determined by a symmetry.

The purpose of this note is to describe the results of [13], where we show that the twist construction can be used to interpret this so-called hyperKähler-quaternionic Kähler correspondence at to prove that there is only one degree of freedom this construction. We then indicate how the computational framework of the twist construction may be applied to understand some of the basic examples of the c-map.

2 Twist constructions

The twist construction [16, 17] associates to a manifold with a circle action a new space of the same dimension with a distinguished vector field.

Suppose M is manifold of dimension n . Let X be a vector field on M that generates a circle action. A twist W of M is specified as a quotient $W = P/\langle X' \rangle$ of a principal S^1 -bundle $P \rightarrow M$ by a lift X' of X . It thus fits in to a double fibration

$$M \xleftarrow{\pi_M} P \xrightarrow{\pi_W} W.$$

If $H^2(M, \mathbb{Z})$ has no torsion, the bundle P is specified by the curvature form F of a connection one-form $\theta \in \Omega^1(P)$, given by $\pi_M^* F = d\theta$. We let $\mathcal{H} = \ker \theta$ be the corresponding horizontal distribution on P . Lifts X' of X that preserve θ and the principal vector field Y are given by

$$X' = X^\theta + (\pi_M^* a)Y,$$

where $X^\theta \in \mathcal{H}$ is the horizontal lift of X with respect to θ and $a \in C^\infty(M)$ is a Hamiltonian function satisfying

$$da = -X \lrcorner F. \quad (1)$$

This requires that F is preserved by X . The *twist* $W := P/\langle X' \rangle$ then admits a circle action generated by $(\pi_W)_* Y$.

This essentials of this set-up are specified by the *twist data* (M, X, F, a) with $X \in \mathfrak{X}(M)$ generating a circle action, $F \in \Omega_{\mathbb{Z}}^2(M)^X$ an X -invariant closed two-form with integral periods and a satisfying (1).

Provided a is non-zero, invariant tensors on M may be transferred to W as follows. Note that at $p \in P$, the projections π_M and π_W induce isomorphisms $T_{\pi_M(p)} M \cong \mathcal{H}_p \cong T_{\pi_W(p)} W$. Thus given $p \in \pi_M^{-1}(q)$, a tensor α_q at $q \in M$ induces a

tensor $(\alpha_W)_{q'}$ at $q' = \pi_W(p) \in W$. The tensor α_W is well-defined if α is preserved by X . We then say that α and α_W are \mathcal{H} -related and write

$$\alpha \sim_{\mathcal{H}} \alpha_W.$$

The two most important computational facts for \mathcal{H} -related tensors are:

Property 1. for $\alpha \in \Omega^p(M)^X$ an invariant p -form, the exterior differential of α_W is given by

$$d\alpha_W \sim_{\mathcal{H}} d_W \alpha := d\alpha - \frac{1}{a} F \wedge X \lrcorner \alpha. \quad (2)$$

Property 2. for an invariant complex structure I on M that is \mathcal{H} -related to an almost complex structure I_W on W , we have

$$I_W \text{ is integrable if and only if } F \text{ is of type } (1, 1) \text{ for } I.$$

Recall that $F \in \Omega^2(M)$ is of type $(1, 1)$ if $F(IA, IB) = F(A, B)$ for all $A, B \in TM$.

These facts show that geometric properties of the twist are determined by the twist data.

Example 1. A basic example of the twist construction is provided by $M = \mathbb{C}P(n) \times T^2$. This is a Kähler manifold as a product. Suppose X generates one of the circle factors of $T^2 = S^1 \times S^1$. Taking F to be the Fubini-Study two-form on $\mathbb{C}P(n)$, we have $X \lrcorner F = 0$, so can take $a \equiv 1$. Then $P = S^{2n+1} \times T^2$ and the twist is $W = S^{2n+1} \times S^1$. As F is type $(1, 1)$ we have that W is a complex manifold. However W is compact and $b_2(W) = 0$, so W can not be Kähler.

3 Elementary deformations of hyperKähler metrics

As formula (2) indicates, the twist of a closed differential form is rarely closed. In a given geometric situation it is therefore interesting to adjust the geometric data before performing a twist.

We wish to work with hyperKähler manifolds. These are (pseudo-)Riemannian manifolds (M, g) with almost complex structures I, J and K such that

1. $IJ = K = -JI$,
2. g is Hermitian with respect to I, J and K ,
3. the two-forms $\omega_I = g(I, \cdot)$, ω_J and ω_K are closed:

$$d\omega_I = 0 = d\omega_J = d\omega_K.$$

By Hitchin [10] the last condition implies that I, J and K are integrable. The restricted holonomy is then a subgroup of $Sp(n)$, where $\dim M = 4n$, and the metric is Ricci-flat. The triples (g, I, ω_I) , etc., are then Kähler structures on M .

Let X be a symmetry of a hyperKähler structure (M, g, I, J, K) , but which we mean that X is an isometry that preserves the linear span $\langle I, J, K \rangle$ of $I, J, K \in \text{End}(TM)$. The vector field X induces four one-forms on M given by

$$\alpha_0 = g(X, \cdot), \quad \alpha_I = I\alpha_0 = -\alpha(I\cdot), \quad \alpha_J = J\alpha_0, \quad \alpha_K = K\alpha_0.$$

We then define

$$g_\alpha = \alpha_0^2 + \alpha_I^2 + \alpha_J^2 + \alpha_K^2.$$

When X is not null, g_α is positive semi-definite proportional to the restriction of g to $\mathbb{H}X = \langle X, IX, JX, KX \rangle$.

Definition 1. An *elementary deformation* of a hyperKähler metric g with respect to a symmetry X is a metric of the form

$$g^N = fg + hg_\alpha$$

with f and h smooth functions on M .

This is thus more general than a conformal change of g .

As I, J and K are parallel, we have that X acts as a linear transformation on $\mathbb{R}^3 = \langle I, J, K \rangle$. It preserves the algebraic relations, so acts as an element of $\mathfrak{so}(3)$. As $\mathfrak{so}(3)$ has rank one, it follows that the action is either trivial or conjugate a circle action fixing I and mapping J to K . By relabelling the complex structures and normalising X we may thus assume in this latter case that

$$L_X I = 0 \quad \text{and} \quad L_X J = K. \quad (3)$$

An isometry X satisfying (3) will be called *rotating*.

For a rotating symmetry, we have $d\alpha_I = 0$, $d\alpha_J = \omega_K$ and $d\alpha_0 = G - \omega_I$, where $G \in \Omega^2(M)$ is a two-form that is of type $(1, 1)$ for I, J and K . As α_I is closed, we may pass to a cover of M and write $\alpha_I = d\mu$ for a smooth map $\mu : M \rightarrow \mathbb{R}$. The function μ is a Kähler moment map for X with respect to (g, ω_I) .

4 The hyperKähler-quaternionic Kähler correspondence

Suppose (M, g, I, J, K) is hyperKähler with rotating symmetry X with Kähler moment map μ . Then X does not preserve ω_J or ω_K , but the four-form

$$\Omega = \omega_I^2 + \omega_J^2 + \omega_K^2 \quad (4)$$

is invariant and closed.

If W is manifold of dimension at least 8 with a four-form Ω^W pointwise linearly equivalent to (4), then W has an almost quaternion-Hermitian structure (g_W, \mathcal{G}) , where $\mathcal{G} \subset \text{End}(TW)$ is a three-dimensional subbundle with a local basis (I_W, J_W, K_W) of almost complex structures for which g_W is Hermitian and with

$I_W J_W = K_W = -J_W I_W$. Such a structure is *quaternionic Kähler* if Ω^W is parallel with respect to the Levi-Civita connection of g_W . It follows that g_W is Einstein [14, 5]. If $\dim W \geq 12$, then to obtain quaternionic Kähler it is sufficient that $d\Omega^W = 0$ [15]. For $\dim W = 8$, one can work with the local two-forms $\omega^W = (\omega_I^W, \omega_J^W, \omega_K^W)$ and quaternionic Kähler is then equivalent to the existence of a local connection form $A \in \Omega^1(\mathfrak{so}(3))$ such that $d\omega^W = A \wedge \omega^W$.

The behaviour of the exterior derivative under the twist is given by (2), so from the above remarks we may determine whether a twist is quaternionic Kähler by working on M .

Theorem 1 [13]. *Let (M, g, I, J, K) be a hyperKähler manifold with non-null rotating symmetry X and Kähler moment map μ . If $\dim M \geq 8$ then, up to homothety, the only twists of elementary deformations $g^N = fg + hg_\alpha$ of g that are quaternionic Kähler have*

$$g^N = \frac{1}{(\mu - c)^2} g_\alpha - \frac{1}{\mu - c} g \quad (5)$$

for some constant c . Furthermore the corresponding twist data is given by

$$F = kG = k(d\alpha_0 + \omega_I), \quad a = k(g(X, X) - \mu + c),$$

for some constant k .

The method of proof is first to impose the quaternionic Kähler condition on an arbitrary twist of Ω^N , the four-form associated to g^N via I, J and K , and to decompose these equations in type components relative to $\mathbb{H}X$ and its orthogonal complement. From this one deduces that f and h are functions of μ and that $h = f'$. Then we consider the equation $da = -X \lrcorner F$ and determine the twist function a . Finally, we investigate the condition $dF = 0$, which provides an ordinary differential equation for f .

From the Theorem, it follows that the constructions provided in [9, 11, 2] of quaternionic Kähler metrics from hyperKähler metrics with rotating circle symmetry agree.

Example 2. We consider $\mathbb{H}^{p,q} = \mathbb{R}^{4p,4q}$, $n = p + q$, with its flat hyperKähler metric

$$g = \sum_{i=1}^n \varepsilon_i (dx_i^2 + dy_i^2 + du_i^2 + dv_i^2)$$

with $\varepsilon_i = +1$, for $i \leq p$, and $\varepsilon_i = -1$, for $i > p$, and Kähler two-forms

$$\begin{aligned} \omega_I &= \sum_{i=1}^n \varepsilon_i (dx_i \wedge dy_i - du_i \wedge dv_i), & \omega_J &= \sum_{i=1}^n \varepsilon_i (du_i \wedge dx_i + dv_i \wedge dy_i), \\ \omega_K &= \sum_{i=1}^n \varepsilon_i (du_i \wedge dy_i - dv_i \wedge dx_i). \end{aligned}$$

If X is a rotating circle symmetry then it is an element of $\mathfrak{sp}(p, q) + \mathfrak{u}(1)$, but lies in a maximal compact subgroup, so is conjugate to

$$X = \sum_{i=1}^n \left(\frac{1}{2} - \lambda_i \right) \left(y_i \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial y_i} \right) - \left(\frac{1}{2} + \lambda_i \right) \left(v_i \frac{\partial}{\partial u_i} - u_i \frac{\partial}{\partial v_i} \right)$$

for some $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. For X to be non-null, we must have $\sum_{i=1}^n \varepsilon_i \lambda_i^2 \neq 0$. This vector field has $d\alpha_0 = d(g(X, \cdot)) = G - \omega_I$ with

$$G = 2 \sum_{i=1}^n \varepsilon_i \lambda_i (dx_i \wedge dy_i + du_i \wedge dv_i)$$

so $G = d\beta$, where $\beta = \sum_{i=1}^n \varepsilon_i \lambda_i (-y_i dx_i + x_i dy_i - v_i du_i + u_i dv_i)$.

The twisting form F is equal to a multiple of $G = d\beta$, so is exact and the twist bundle is trivial $P = \mathbb{H}^n \times S^1$. Let us take $F = G$. The connection one-form may be written as $\theta = \beta + d\tau$, where $\partial/\partial\tau$ generates the principal S^1 -action. The horizontal lift X^θ of X to P is then

$$X^\theta = X - \beta(X) \frac{\partial}{\partial\tau}.$$

Direct calculation shows that $d(\beta(X)) = -X \lrcorner F$, so the twist function is $a = \beta(X) + c$ and the twist is the quotient of P by $X' = X + c \frac{\partial}{\partial\tau}$. Thus the twist is

$$W = (\mathbb{H} \times S^1) / \langle X + c \frac{\partial}{\partial\tau} \rangle.$$

This will be an orbifold if λ_i and c are integers. It is smooth when they are pairwise co-prime.

The theorem says that W is equipped with a quaternionic Kähler metric \mathcal{H} -related to g^N in equation (5), whenever this is non-degenerate. The function μ is given by

$$\mu = \frac{1}{2} \sum_{i=1}^n \left(\frac{1}{2} - \lambda_i \right) (x_i^2 + y_i^2) + \left(\frac{1}{2} + \lambda_i \right) (u_i^2 + v_i^2).$$

The metric g^N has two contributions to its signature. On the quaternionic span $\mathbb{H}X$ of X , the sign is that of $(\|X\|^2 - \mu + c)\|X\|^2/(\mu - c)^2$, orthogonal to $\mathbb{H}X$ the original metric is multiplied by $-\|X\|^2/(\mu - c)$. Thus up to overall sign g^N has quaternionic signature that is either $(p+1, q-1)$, (p, q) or $(p-1, q+1)$. It is degenerate on the sets $(\|X\|^2 = 0)$, i.e., where X is null, and on $(\|X\|^2 - \mu + c = 0)$, which is the set where the twist function a vanishes. The metric may also blow-up on $(\mu = c)$.

5 Application to the c-map

The c-map is a construction introduced by Cecotti et al. [6]. It starts with a so-called projective special Kähler manifold of dimension $2n$ and produces a quaternionic Kähler manifold of dimension $4n+4$. Explicit local expressions for the resulting metrics were provided by Ferrara and Sabharwal [7]. Recently Alekseevsky et al. [3, 1] have shown that the hyperKähler-quaternionic Kähler correspondence re-

produces the quaternionic Kähler metrics of the c-map. In particular, this means that one may obtain all the known examples homogeneous (positive definite) quaternionic Kähler of negative scalar curvature, and their work is also beginning to produce new examples of complete quaternionic Kähler metrics.

Given the wide generality of the twist construction, is useful to understand how such homogeneous examples may arise. To be concrete let us consider the real hyperbolic space $\mathbb{R}H(2)$ as a solvable Lie group with Kähler metric of constant curvature. This has a global basis $\{a, b\}$ of one forms, such that $da = 0$ and $db = -\lambda \alpha \wedge b$, for some constant λ depending on the scalar curvature. For this to be a projective special Kähler manifold, we need to consider a certain cone metric and show that it admits a flat symplectic connection of special Kähler type, as described by Freed [8].

Let C_0 be a circle bundle over $\mathbb{R}H(2)$ with connection one-form ϕ whose curvature is $2a \wedge b$. Pulling a and b back to $C = \mathbb{R}_{>0} \times C_0$, the cone geometry is

$$g_C = t^2(a^2 + b^2 - \phi^2) - dt^2, \quad \omega_C = t^2 a \wedge b - t \phi \wedge dt,$$

a Kähler metric of signature $(2, 2)$. It has a symmetry X generated by the principal action on C_0 .

Locally, one can show that this admits a special Kähler connection if and only if λ^2 is 4 or $4/3$. In case $\lambda^2 = 4$, the special connection agrees with the Levi-Civita connection of g_C . In both cases, using the cotangent trivialisation $(\hat{a}, \hat{b}, \hat{\phi}, \hat{\psi}) = (ta, tb, t\phi, dt)$, one may construct a hyperKähler metric of signature $(4, 4)$ on $H = T^*C$ of the form $g_H = \hat{a}^2 + \hat{b}^2 - \hat{\phi}^2 - \hat{\psi}^2 + \hat{A}^2 + \hat{B}^2 - \hat{\Phi}^2 - \hat{\Psi}^2$. Indeed the flat connection gives $TH = V^* \oplus V$, with $V \cong TM$. This is the rigid c-map, see Freed [8]. The Kähler forms ω_J and ω_K are just the real and imaginary parts of the standard complex symplectic two-form on $H = T^*C$.

Horizontally lifting the symmetry of X of C to $H = T^*C$ using the flat connection, one obtains a rotating symmetry \tilde{X} of the hyperKähler structure. Note that the symmetry X does not preserve the flat connection, and it rotates the quadruple $\tilde{\delta} = (\hat{A}, \hat{B}, \hat{\Phi}, \hat{\Psi})$. The twist data for this lifted action is given by the curvature form

$$F = -\hat{a} \wedge \hat{b} + \hat{\phi} \wedge \hat{\psi} - \hat{A} \wedge \hat{B} + \hat{\Phi} \wedge \hat{\Psi}$$

and twist function $-t^2/2 + c$. The curvature form is exact, and so we may proceed much as in Example 2.

In particular, we have a coordinate τ on S^1 in $P = H \times S^1$. With $c = 0$ the twist is then diffeomorphic to $(H/\langle \tilde{X} \rangle) \times S^1$. We may use τ to define a new quadruple $\delta = \tilde{\delta} \exp(\mathbf{i}\tau)$, where $\mathbf{i} = \text{diag}(\mathbf{i}_2, \mathbf{i}_2)$ with $\mathbf{i}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Now using (2) one may show that the structure functions of the coframe \mathcal{H} -related to $(\hat{a}, \hat{b}, \hat{\phi}, \hat{\psi}, \delta_1, \delta_2, \delta_3, \delta_4)$ are constants, so these define a dual basis for a Lie algebra. The metric g^N is seen to be positive definite, complete and has constant coefficients in this coframe, so the resulting quaternionic Kähler metric on W is complete. It follows that the universal cover of W is a Lie group G and that the metric on W pulls back to a left-invariant metric. We have $W = G/\mathbb{Z}$ and knowing the structure constants we may identify G as the solvable Lie groups that act transitively on the non-compact symmetric spaces

$\mathrm{Gr}_2(\mathbb{C}^{2,2})$ for $\lambda^2 = 4$ or $G_2^*/SO(4)$ for $\lambda^2 = 4/3$. This provides a global verification of the main example of Ferrara and Sabharwal [7].

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